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On structures of weak interlaced bilattices

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Abstract

We study fundamental properties of weak interlaced bilattices and show that

1. For any bounded lattice L , there exists an interlaced bilattice \mathcal{B} such that $\mathcal{K}(L) \cong \text{Cons}(\mathcal{B})$;
2. For any interlaced bilattice \mathcal{B} with negation, there exists a lattice L such that $\mathcal{K}(L) \cong \text{Cons}(\mathcal{B})$.

1 Introduction

It is well-known the Kleene's 3-valued logic in the field of multiple-valued logics. The logic has three values *false*, *true*, and \perp (*unknown*) as truth values. These values have two informal orderings concerning "amount of knowledge" and "degree of truth". For example, if we think of a certain proposition such as *Riemann's conjecture* assigned \perp as truth value, then it is possible that we can conclude the truth value of the proposition as *true* or *false* with increasing knowledge. Thus in the ordering of knowledge, \perp is smaller than *true* and *false*. A sentence with \perp is between *false* and *true* in the ordering of degree of truth. In this way it can be considered that the three valued logic has two orderings. Belnap ([2]), Ginsberg([5]), and others proposed concept of a *bilattice* which has two orderings and proved some fundamental results ([1, 3, 4]). It is shown by Fitting ([3]) that bilattices can give a uniform semantics for many languages of logic programming. Since then the theory of bilattices is a hot research field.

On the other hand, as in *Fuzzy logics*, a truth value can be taken as a closed interval $[a, b]$. Let L be a lattice and $\mathcal{K}(L)$ be the set of all closed intervals of L . In this case we also define two orderings. For $[a, b], [c, d] \in \mathcal{K}(L)$, if $[a, b] \subseteq [c, d]$ then the knowledge in $[a, b]$ is greater than that in $[c, d]$. Thus we set $[a, b] \sqsubseteq_k [c, d]$ if $[a, b] \subseteq [c, d]$. Likewise we also define $[a, b] \sqsubseteq_t [c, d]$ if $a \leq c$ and $b \leq d$, because $[c, d]$ is greater than $[a, b]$ in the ordering degree of truth. The structure $\mathcal{K}(L) = \langle \mathcal{K}(L), \sqsubseteq_t, \sqsubseteq_k \rangle$ which precise definition is given below has the property of *weak interlaced bilattice*.

In [3, 4], Fitting, Font and Moussavi have investigated the structure of $\mathcal{K}(L)$ and proved some results :

1. If L is a bounded lattice, then $\mathcal{K}(L)$ is a weak interlaced bilattice ([4]) ;
2. If L is a complete lattice with an *involution*, then $\mathcal{K}(L) \cong \text{Cons}(\mathcal{B})$, where $\text{Cons}(\mathcal{B})$ is the set of all *consistent* elements of an *interlaced bilattice* \mathcal{B} with *negation* and *conflation* ([3]) ;

3. If \mathcal{B} is a distributive bilattice with commutative negation and conflation, then $\text{Cons}(\mathcal{B}) \cong \mathcal{K}(L)$ for some complete distributive lattice L ([3]).

Now it is natural to ask the following questions :

- Q1** Is there a lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$ for every weak interlaced bilattice \mathcal{W} ?
- Q2** Is there an interlaced bilattice \mathcal{B} such that $\mathcal{K}(L) \cong \text{Cons}(\mathcal{B})$ for every bounded lattice L ?
- Q3** Is there a lattice L such that $\mathcal{K}(L) \cong \text{Cons}(\mathcal{B})$ for every interlaced bilattice with negation \mathcal{B} ?

In the following, we study properties of $\mathcal{K}(L)$ and answer the questions above.

2 Definition of $\mathcal{K}(L)$

We define a structure $\mathcal{K}(L)$ for any lattice L . Let $L = (L, \leq)$ be a lattice and $\mathcal{K}(L)$ be the set of all closed intervals of L , that is,

$$\begin{aligned} K(L) &= \{[a, b] \mid a \leq b, a, b \in L\} \\ [a, b] &= \{x \mid a \leq x \leq b\}. \end{aligned}$$

For any $[a, b], [c, d] \in K(L)$, we define two orderings $\sqsubseteq_t, \sqsubseteq_k$ on $K(L)$ as follows :

$$\begin{aligned} [a, b] \sqsubseteq_t [c, d] &\iff a \leq c, b \leq d \\ [a, b] \sqsubseteq_k [c, d] &\iff a \leq c, b \geq d \end{aligned}$$

We set $\mathcal{K}(L) = \langle K(L), \sqsubseteq_t, \sqsubseteq_k \rangle$. It is obvious from definition that $[0, 0]$ ($[1, 1]$) is the minimum (maximum) element with respect to \sqsubseteq_t . On the other hand, while $[0, 1]$ is the minimum element, there is no maximum element with respect to the ordering \sqsubseteq_k . This means that $\mathcal{K}(L)$ is a lattice with respect to \sqsubseteq_t and is a semi-lattice concerning \sqsubseteq_k . Four operators $\sqcap_t, \sqcup_t, \sqcap_k, \sqcup_k$ are defined by

$$\begin{aligned} \inf_{\sqsubseteq_t} \{a, b\} &= a \sqcap_t b \\ \sup_{\sqsubseteq_t} \{a, b\} &= a \sqcup_t b \\ \inf_{\sqsubseteq_k} \{a, b\} &= a \sqcap_k b \\ \sup_{\sqsubseteq_k} \{a, b\} &= a \sqcap_k b \quad (\text{if it is defined}) \end{aligned}$$

A relational system $\langle B, \leq_t, \leq_k \rangle$ is called an *interlaced bilattice* if it satisfies

1. B is a non-empty set
2. $\langle B, \leq_t \rangle, \langle B, \leq_k \rangle$ are bounded lattices and satisfy

- (a) $x \leq_t y \implies x \otimes z \leq_t y \otimes z, x \oplus z \leq_t y \oplus z$
- (b) $x \leq_k y \implies x \wedge z \leq_k y \wedge z, x \vee z \leq_k y \vee z$

By $0(1)$, we mean the minimum (maximum) element with respect to the ordering \leq_t . We also denote by $\perp(\top)$ the minimum (maximum) element concerning \leq_k .

Any interlaced bilattice is called *distributive* when it satisfies

$$x \circ (y \bullet z) = (x \circ z) \bullet (y \circ z)$$

for $\circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$. This means twelve equations such as

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$$

\vdots

A map \neg from B into itself is called a *negation* if

$$x \leq_t y \implies \neg y \leq_t \neg x$$

$$x \leq_k y \implies \neg x \leq_k \neg y$$

$$\neg \neg x = x.$$

For lattices $L_1 = \langle L_1, \wedge_1, \vee_1 \rangle$ and $L_2 = \langle L_2, \wedge_2, \vee_2 \rangle$, we define operations $\wedge, \vee, \otimes, \oplus$ on the product $L_1 \times L_2$: For $(a, b), (c, d) \in L_1 \times L_2$,

$$(a, b) \wedge (c, d) = (a \wedge_1 c, b \vee_2 d)$$

$$(a, b) \vee (c, d) = (a \vee_1 c, b \wedge_2 d)$$

$$(a, b) \otimes (c, d) = (a \wedge_1 c, b \wedge_2 d)$$

$$(a, b) \oplus (c, d) = (a \vee_1 c, b \vee_2 d).$$

The structure $L_1 \odot L_2 = \langle L_1 \times L_2, \wedge, \vee, \otimes, \oplus \rangle$ is called a *Ginsberg product*. There are some fundamental results about the structure :

Proposition 1 (Fitting). *If L_1, L_2 are bounded lattices then the Ginsberg product $L_1 \odot L_2 = \langle L_1 \times L_2, \wedge, \vee, \otimes, \oplus \rangle$ is an interlaced bilattice. Especially, $L \odot L$ is an interlaced bilattice with negation \neg , where \neg is defined by $\neg(a, b) = (b, a)$.*

It is proved that the converse holds by Avron ([1]).

Proposition 2 (Avron). *For any interlaced bilattice \mathcal{B} , there are bounded lattices L_1, L_2 such that $\mathcal{B} \cong L_1 \odot L_2$. In particular, for any interlaced bilattice \mathcal{B} with negation, there is a bounded lattice L such that $\mathcal{B} \cong L \odot L$.*

It is clear from definition that orderings $\sqsubseteq_t, \sqsubseteq_k$ on $\mathcal{K}(L)$ are the same as \leq_t, \leq_k on Ginsberg product $L \odot L$, respectively :

$$\sqsubseteq_t \text{ in } \mathcal{K}(L) \iff \leq_t \text{ in } L \odot L$$

$$\sqsubseteq_k \text{ in } \mathcal{K}(L) \iff \leq_k \text{ in } L \odot L$$

Next we give a definition of a *weak interlaced bilattice* according to Font ([4]). A structure $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ is called a *weak interlaced bilattice* if

1. $\langle W, \leq_t \rangle$: lattice
2. $\langle W, \leq_k \rangle$: meet semilattice
3. $a \leq_k b, c \leq_k d \implies a \wedge c \leq_k b \wedge d, a \vee c \leq_k b \vee d$
4. $a \leq_t b, c \leq_t d \implies a \otimes c \leq_t b \otimes d,$
5. $a \leq_t b, c \leq_t d \implies a \oplus c \leq_t b \oplus d$ if $a \oplus c$ and $b \oplus d$ exist.

3 Properties of weak interlaced bilattices

For any weak interlaced bilattice \mathcal{W} , if we define

$$\begin{aligned} L_1 &= \{x \in \mathcal{W} \mid x \leq_k 0\} = [\perp, 0]_k \\ L_2 &= \{x \in \mathcal{W} \mid x \leq_k 1\} = [\perp, 1]_k, \end{aligned}$$

then we have

Proposition 3.

$$\begin{aligned} L_1 &= [\perp, 0]_k = [0, \perp]_t \\ L_2 &= [\perp, 1]_k = [\perp, 1]_t \end{aligned}$$

Proof. Let $x \in [\perp, 0]_k$. Since $\perp \leq_k x \leq_k 0$, we have $\perp \vee \perp \leq_k x \vee \perp \leq_k 0 \vee \perp$ by definition of weak interlaced bilattice. From $\perp \vee \perp = 0 \vee \perp = \perp$, it follows that $x \vee \perp = \perp$ and hence that $x \leq_t \perp$. This means $[\perp, 0]_k \subseteq [0, \perp]_t$.

Conversely, suppose $x \in [0, \perp]_t$. If we put $u = 0 \otimes x$, then it is clear that $u \leq_k 0$ and $u \leq_k x$. Since $0 \leq_t x$, we have $0 \otimes x \leq_t x \otimes x = x$ and hence $u \leq_t x$. It follows from $\perp \leq_k u$ that $x \wedge \perp \leq_k x \wedge u$. Since $x \leq_t \perp$, we also have $x \wedge \perp = x$. On the other hand, since $u \leq_t x$, we get $u \wedge x = u$. These imply that $x \leq_k u$ and hence that $x = u$. Thus we have $x \leq_k 0$. Namely, we have $[0, \perp]_t \subseteq [\perp, 0]_k$.

The second equation can be proved similarly. □

The result implies that L_1 and L_2 are lattices with ordering \leq_1 and \leq_2 in \mathcal{B} , respectively, where \leq_1 and \leq_2 are defined by

$$\begin{aligned} \leq_1 &= \leq_t = \geq_k \\ \leq_2 &= \leq_t = \leq_k \end{aligned}$$

Thus we can consider the Ginsberg product $L_1 \odot L_2$, which becomes an *interlaced bilattice*. Moreover we can prove

Proposition 4. Let \mathcal{W} be any weak interlaced bilattice. For any $x \in \mathcal{W}$, we have

$$x = (x \otimes 0) \oplus (x \otimes 1) = (x \wedge \perp) \vee (x \vee \perp)$$

Proof. See Avron [1] Cor.3.8 □

Now we investigate a relation between a weak interlaced bilattice \mathcal{W} and an interlaced bilattice $L_1 \odot L_2$ constructed by \mathcal{W} .

Lemma 1. A map $\xi : \mathcal{W} \rightarrow L_1 \times L_2$ defined by $\xi(x) = (x \otimes 1, x \otimes 0) = (x \vee \perp, x \wedge \perp)$ is an embedding.

This means that

Theorem 1. Any weak interlaced bilattice can be embedded into an interlaced bilattice.

As to the question Q1, we can give a negative answer by presenting a counter example. Let \mathcal{W} be the set $\{0, 1, a, b, \perp, 1\}$ such that $0 \leq_t a \leq_t \perp \leq_t b \leq_t 1$ and $\perp \leq_k a \leq_k 0, \perp \leq_k b \leq_k 1$. It is obvious that \mathcal{W} is a weak interlaced bilattice. Suppose that there is a lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$. If $|L| \geq 3$, then there exists an element $a \in L$ such that $0 < a < 1$. For that element we have $[0, 0], [0, a], [0, 1], [a, 1], [a, a], [1, 1] \in \mathcal{K}(L)$ and $|\mathcal{K}(L)| \geq 6$. Since $|\mathcal{W}| = 5$, it must be $|L| \leq 2$. But, in this case, we have $|\mathcal{K}(L)| \leq 3$. This means that there is no lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$.

4 Characterization of $\mathcal{K}(L)$

In this section we consider the properties of $\mathcal{K}(L)$ for a bounded lattice L . Let $\mathcal{I}(L) = \{(a, b) \mid a \leq b\} \subseteq L \times L$. It is clear that $\mathcal{I}(L)$ is closed under the operations \wedge , \otimes and \oplus but not closed under \vee . If we define a map $\eta : \mathcal{K}(L) \rightarrow \mathcal{I}(L)$ by $\eta([a, b]) = (a, b)$, then we can prove that

Lemma 2. $\eta : \mathcal{K}(L) \rightarrow \mathcal{I}(L) : \text{bijection and}$

$$\begin{aligned} \eta([a, b] \sqcap_t [c, d]) &= \eta([a, b]) \otimes \eta([c, d]) \\ \eta([a, b] \sqcup_t [c, d]) &= \eta([a, b]) \oplus \eta([c, d]) \\ \eta([a, b] \sqcap_k [c, d]) &= \eta([a, b]) \wedge \eta([c, d]) \\ \text{if } [a, b] \oplus [c, d] \text{ exists } \eta([a, b] \sqcup_k [c, d]) &= \eta([a, b]) \vee \eta([c, d]) \end{aligned}$$

We call the map η a *t-k dual isomorphism* and identify the isomorphism with the t-k dual isomorphism, that is,

$$\mathcal{K}(L) \cong \mathcal{I}(L) \subseteq L \odot L$$

In any interlaced bilattice $L \odot L$, the negation \neg is defined by

$$\neg(a, b) = (b, a).$$

An element (a, b) in $L \odot L$ is called *consistent* when it satisfies $(a, b) \leq_t \neg(a, b)$, that is,

$$(a, b) : \text{consistent} \iff a \leq b$$

If we denote by $\text{Cons}(\mathcal{B})$ the set of all consistent elements of an interlaced bilattice \mathcal{B} , since $\text{Cons}(L \odot L) = \mathcal{I}(L)$, then we have

Theorem 2. For any bounded lattice L ,
 $\mathcal{K}(L) \cong \mathcal{I}(L) = \text{Cons}(L \odot L)$

This means that we can answer the question Q2 as Yes. Moreover, for the structure $\mathcal{I}(L)$, we can show

Theorem 3. $\mathcal{I}(L)$ is the weak interlaced bilattice generated by $\Delta = \{(a, a) \mid a \in L\}$.

Proof. Let \mathcal{W} be any weak interlaced bilattice such that $\Delta \subseteq \mathcal{W}$. For every element $(a, b) \in \mathcal{I}(L)$ ($a \leq b$), since $(a, a), (b, b) \in \Delta \subseteq \mathcal{W}$, we have $(a, a) \wedge (b, b) = (a \wedge b, a \vee b) = (a, b) \in \mathcal{W}$. Thus $\mathcal{I}(L) \subseteq \mathcal{W}$. \square

As to the question Q3, let \mathcal{B} be any interlaced bilattice with negation. Since there is a lattice L such that $\mathcal{B} \cong L \odot L$, by identifying \mathcal{B} with $L \odot L$, we have $\text{Cons}(\mathcal{B}) = \text{Cons}(L \odot L) = \mathcal{I}(L) \cong \mathcal{K}(L)$. This means that

Theorem 4. For any interlaced bilattice \mathcal{B} with negation, there is a lattice L such that

$$\text{Cons}(\mathcal{B}) \cong \mathcal{K}(L)$$

Therefore we can answer the question Q3 as Yes.

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